

The standard definition also leads to the *hypergame paradox* (Zwicker, 1987). In hypergame, player 1 chooses any game at all for player 2 to play. The problem is that players can keep choosing hypergame so that the game never terminates. So no formal system of games can include hypergame, which is counterintuitive. In practice, theorists define a class of non-terminating *loopy games* whose graphical representations have loops. These, in turn, open up to *transfinite games*.

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Applications of combinatorial games are few, but do exist. *Lexicodes* are a type of error-correcting code where “code words correspond to winning positions in certain impartial games,” so that if the underlying game is known, it can be solved to reveal the message (Milvang-Jensen, 2000: 91). Several well-known codes (e.g. Hamming codes) are cases of lexicodes. Games have also been used in computer science to describe types of data structures (Burke, 2015).⁽¹⁰¹⁰⁾

By far the most interesting (potential) application is *surreal analysis*. Like real analysis, the aim is to provide foundations for calculus on surreal numbers. Many functions used in physics diverge to infinity, giving nonsensical answers to well-posed questions; hence mathematicians hope surreal analysis will let them rigorously work *with* these infinities. The main obstacle is that no one has yet been able to satisfactorily define integration (Matthews, 1995).

Perhaps someday all of these research programmes may converge to a new foundation for science—a glass bead game.

Nevertheless, the proposition “All the world’s a game” is not well-formed. We know that all combinatorial games can be reduced to tuples of the void, and like surreal numbers, they do not *exist*. If all the world is a game, then there is no ‘world’ to be a game. So even if everything is fully describable in game theoretic terms, the ‘is’ remains undefined.

Thus: Existence is itself played by an *arché-game* prior to difference and ontology. We call this the One-in-One.

by Graham Jonas

Notes:

(1): Practically, we write the surreals as $\{L|R\}$, with L and R as sets of numbers. Surreal numbers obey the rule: no member of L is \geq any member of R, i.e. $\ell \not\geq r (\forall \ell \in L)(\forall r \in R)$. For $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$, we say $x \leq y$ iff no x^L in X^L is $\geq y$ and no y^R in Y^R is $\leq x$. Then $x = y$ iff $x \geq y$ and $x \leq y$. Define addition as $x + y = \{X^L + y, x + Y^L | X^R + y, x + Y^R\}$, e.g. $\frac{1}{2} + \frac{1}{2} = \{0|1\} + \{0|1\} = \{0 + \frac{1}{2}, \frac{1}{2} + 0 | 1 + \frac{1}{2}, 1 + \frac{1}{2}\} = \{\frac{1}{2}|\frac{3}{2}\} = 1$. For the last equality, note: $\frac{1}{2} \not\geq 1$ and $\frac{3}{2} \not\leq \frac{1}{2} + \frac{1}{2}$, so $\{\frac{1}{2}|\frac{3}{2}\} \leq 1$. Next, writing $1 = \{0|\}$, note: $0 \not\geq \frac{1}{2} + \frac{1}{2}$ and $\frac{3}{2} \not\leq 1$, so $1 \leq \{\frac{1}{2}|\frac{3}{2}\}$. Thus $\{\frac{1}{2}|\frac{3}{2}\} = 1$. For a step-by-step derivation from first principles, see Knuth’s quasi-novel *Surreal Numbers* (1974) which coined their name.

(10): This is from Nowakowski (2009: 7-8). A *group* has a set and an operation combining two elements to get a third. For games, the operation is addition. Games count as a group since they have an identity element ($G + 0 = G$, so G is identical), have an inverse ($-G$), are closed (any $G + H$ is a game), and are associative. Since games are commutative, they are Abelian.

(11): Nim was formalized by Bouton in 1902. The origin of its name is confirmed by Walsh (1953).

(100): As well as nim-sums, one can also do nim-multiplication, making numbers a group theoretic *field* (cf. Ferguson, 2014).

(101): Another way to think about uniqueness is that numbers make a closed system: if $a \oplus b \oplus c = 0 \rightarrow a \oplus b = c$. This can be rearranged for any missing number in the system. Systems of n numbers may be solved similarly.

(110): We’ve only covered normal play, where a player who can’t move loses. An alternate convention is *misère play*: a player who can’t move *wins*. In some simple misère games, such as Nim, misère and normal play are largely similar; yet, in the majority of cases misère games are far harder to solve. Notably, the SG-function does not always apply.

(111): The two proofs are from Ferguson (2014), followed by Albert, Nowakowski & Wolfe’s (2007) notation for numbers.

(1000): This example is from Demaine (2009). The next paragraph on \mathbb{H}_x and \mathbb{H}_x is from Uiterwijk & Barton (2015: 5).

(1001): Another partizan game is Hex, invented by John Nash. It has a rhombus-shaped board with hexagonal spaces. Each player places a stone on some hexagon, and the goal is to form a diagonal line reaching from one end to the other: player 1 uses white stones to form an off-diagonal line (bottom-left to top-right), player 2 wants a black diagonal line. Curiously, draws are impossible in Hex: this is guaranteed by Brouwer’s fixed point theorem, the idea behind Nash equilibrium. Further, Nash showed player 1 has a winning strategy by a ‘strategy-stealing’ argument: if player 2 has an optimal strategy, player 1 can start with an arbitrary move, then copy all of player 2’s moves. So player 1 can ‘steal’ player 2’s optimal strategy, plus have the advantage of an extra stone on the board, and thus can win. Yet, this proof is non-constructive: we know an optimal strategy *exists*, but can’t find it. The proof for Nash equilibrium is similarly non-algorithmic.

(1010): A variant of Nim can even be used to derive Chebyshev’s inequality (Stolarsky, 1991). As well, algorithmic tools can be used to prove that certain combinatorial games are PSPACE-complete, EXPTIME-complete, or EXPSpace-complete.

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TL;DR

- Any number can be written as a tuple of games played by the void (\emptyset) with itself.
- We can also treat games like numbers. In fact, numbers are a subclass of games.
- In Nim, players take stones from piles. To find a winning strategy, we invent *numbers*.
- Sprague-Grundy theorem: any impartial (symmetric) game is equivalent to a number.
- Solutions for partizan (asymmetric) games use various other pseudo-numbers.
- Mathematicians want to use surreal numbers for new foundations of mathematics.
- Surreal numbers, numbers, and pseudo-numbers can all be written as tuples of \emptyset .
- Therefore combinatorial games support the thesis that numbers do not truly 'exist'.
- It is invalid to say "All the world's a game," since games are solely composed of \emptyset .